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Stability of Transient Solution of Moderately Thick Plate by Finite-Difference Method

Tien Yu Tsui*
Army Materials and Mechanics Research Center, Watertown, Mass.

AND

Pin Tong†
Massachusetts Institute of Technology, Cambridge, Mass.

Introduction

The finite-difference method, widely used in the solution of the equation of motion which governs the transient response of structures such as plates and shells, can become unstable unless the ratio of the time mesh to the space mesh satisfies a certain condition. The condition of stability of the finite-difference equation for the transient response of a "thin" flat plate has been given by Leech.[1] This paper is concerned with the same problem for a "moderately thick" plate. The method of stability analysis used here follows that employed in Ref. 1.

Method of Analysis

The governing equation of motion of a moderately thick plate has been derived by Mindlin[2] as follows:

$$\left(\nabla^2 - \frac{\rho}{G'}\frac{\partial^2}{\partial t^2}\right)\left(D\nabla^2 - \frac{\rho h^3}{12}\frac{\partial^2}{\partial t^2}\right) w + \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad (1)$$

where w is the displacement normal to the initial plane of the plate, D the flexural rigidity of the plate, ρ the density of plate material, h the plate thickness, and G' is related to the shear modulus G by $G' = (\pi^2/12)G$. The preceding equation includes the effect of rotatory inertia and transverse shear. When a square space mesh $\Delta x = \Delta y$ is used the corresponding explicit finite-difference equation for Eq. (1) may be written as follows:

$$\begin{aligned}
&[D/(\Delta x)^4](w_{j-2,k} - 8w_{j-1,k} + 20w_{j,k} - 8w_{j+1,k} + \\
&\quad w_{j+2,k} + 2w_{j-1,k+1} - 8w_{j,k+1} + 2w_{j+1,k+1} + \\
&\quad w_{j,k+2} + 2w_{j-1,k-1} - 8w_{j,k-1} + 2w_{j+1,k-1} + \\
&\quad w_{j,k-2})_n - [a/(\Delta x\Delta t)^2][(w_{j+1,k} - 4w_{j,k} + \\
&\quad w_{j-1,k} + w_{j,k+1} + w_{j,k-1})_{n+1} - 2(w_{j+1,k} - \\
&\quad 4w_{j,k} + w_{j-1,k} + w_{j,k+1} + w_{j,k-1})_{n-1} + \\
&\quad (w_{j+1,k} - 4w_{j,k} + w_{j-1,k} + w_{j,k+1} + w_{j,k-1})_{n-1}] + \\
&\quad [b/(\Delta t)^4](w_{n+2} - 4w_{n+1} + 6w_n - 4w_{n-1} + \\
&\quad w_{n-2})_{j,k} + [c/(\Delta t)^2](w_{n+1} - 2w_n + w_{n-1})_{j,k} = 0 \quad (2)
\end{aligned}$$

where $a = \rho h^3/12 \times D\rho/G'$, $b = \rho^2h^3/12G'$, $c = \rho h$. In Eq. (2) j and k denote space mesh stations x and y, respectively, and n denotes the current time t. When the value of w has been determined at all space mesh points for the last two time increments n and $n - 1$, one can directly calculate the solution for the next time increment $n + 1$ from Eq. (2), thus advancing the solution in time. The first two values required to start the whole numerical process are obtained from the initial conditions. If it is assumed in the practical solution of the finite-difference Eq. (2) that the error $\delta(x,y,t)$ is resulted from roundoff, it must satisfy the following equation:

$$\begin{aligned}
&[D/(\Delta x)^4](\delta_{j-2,k} - 8\delta_{j-1,k} + 20\delta_{j,k} - 8\delta_{j+1,k} + \\
&\quad \delta_{j+2,k} + 2\delta_{j-1,k+1} - 8\delta_{j,k+1} + 2\delta_{j+1,k+1} + \\
&\quad \delta_{j,k+2} + 2\delta_{j-1,k-1} - 8\delta_{j,k-1} + 2\delta_{j+1,k-1} + \\
&\quad \delta_{j,k-2})_n - [a/(\Delta x\Delta t)^2][(\delta_{j+1,k} - 4\delta_{j,k} + \\
&\quad \delta_{j-1,k} - \delta_{j,k+1} + \delta_{j,k-1})_{n+1} - 2(\delta_{j+1,k} - \\
&\quad 4\delta_{j,k} + \delta_{j-1,k} + \delta_{j,k+1} + \delta_{j,k-1})_{n-1} + (\delta_{j+1,k} - \\
&\quad 4\delta_{j,k} + \delta_{j-1,k} + \delta_{j,k+1} + \delta_{j,k-1})_{n-1}] + \\
&\quad [b/(\Delta t)^4][\delta_{n+2} - 4\delta_{n+1} + 6\delta_n - 4\delta_{n-1} + \\
&\quad \delta_{n-2})_{j,k} + [c/(\Delta t)^2](\delta_{n+1} - 2\delta_n + \delta_{n-1})_{j,k} = 0 \quad (3)
\end{aligned}$$

Following the procedure employed by Leech,[1] it is assumed that the general term of the Fourier series expansion of the numerical error may be expressed in the form

$$\delta_{j,k,n} = e^{\alpha n\Delta t}\, e^{i\beta j\Delta x}\, e^{i\gamma k\Delta y} \quad (4)$$

Making use of Eqs. (3) and (4) one obtains the following equation:

$$\begin{aligned}
&\left[(\xi)^{1/2} - \frac{1}{(\xi)^{1/2}}\right]^4 + r^2\left[4\left(\frac{a}{b}\right)\left(\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right) + \right. \\
&\quad \left.\left(\frac{c}{b}\right)(\Delta x)^2\right]\left[(\xi)^{1/2} - \frac{1}{(\xi)^{1/2}}\right]^2 + \\
&\quad \frac{16Dr^4}{b}\left(\sin^2\frac{\beta\Delta x}{2} + \sin^2\frac{\gamma\Delta y}{2}\right)^2 = 0 \quad (5)
\end{aligned}$$

where $\xi = e^{\alpha\Delta t}$ and $r = \Delta t/\Delta x$. For convenience in examining the stability of the finite-difference solution, the following symbol is introduced:

$$Z = \left[(\xi)^{1/2} - \frac{1}{(\xi)^{1/2}}\right]^2 \quad (6)$$

In terms of Z, Eq. (5) becomes

$$Z^2 + 2r^2\alpha Z + r^4\beta = 0 \quad (7)$$

where

$$\alpha = 2(a/b)[\sin^2(\beta\Delta x/2) + \sin^2(\gamma\Delta y/2)] + (c/2b)(\Delta x)^2$$
$$\beta = (16D/b)[\sin^2(\beta\Delta x/2) + \sin^2(\gamma\Delta y/2)]^2$$

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* Mechanical Engineer, Structural Mechanics Division, Theoretical and Applied Mechanics Research Laboratory.

† Associate Professor, Department of Aeronautics and Astronautics.

The solution of Eq. (7) is given by

$$Z = [-\alpha \pm (\alpha^2 - \beta)^{1/2}]\, r^2 \tag{8}$$

It can be shown that $\alpha^2 > \beta$. Therefore, Z is a real quantity.

To facilitate the ensuing discussion of stability, Eq. (6) is recast in the following form:

$$\xi^2 - 2[1 + (Z/2)]\xi + 1 = 0 \tag{9}$$

In order for the error not to grow with increasing time, $|\xi|$ must be less than or equal to one. This condition requires that

$$|1 + (Z/2)| \leq 1 \tag{10}$$

Equations (8) and (10) may be combined to yield

$$-4 \leq [-\alpha \pm (\alpha^2 - \beta)^{1/2}]\, r^2 \leq 0 \tag{11}$$

Since the right-hand inequality is always satisfied for all values of r, one needs consider only the left-hand inequality. Furthermore, one needs consider only the conservative case, namely:

$$[\alpha + (\alpha^2 - \beta)^{1/2}]\, r^2 \leq 4$$

which in turr yields

$$\Delta t \leq 2\Delta x/[\alpha + (\alpha^2 - \beta)^{1/2}]^{1/2} \tag{12}$$

In order to assure a stable numerical calculation, it is useful to determine an upper bound for r, or the largest time increment permissible for a selected space mesh size. This is accomplished by substituting the maximum possible value of α in Eq. (12). The following equation is obtained as a result:

$$\Delta t \leq \frac{2\Delta x}{(4(a/b) + (c/2b)(\Delta x)^2 + \{[4(a/b) + (c/2b)(\Delta x)^2]^2 - 64D/b\}^{1/2})^{1/2}} \tag{13}$$

If the value of β in Eq. (12) is taken to be zero, the following simple result is obtained:

$$\Delta t \leq \frac{\Delta x}{[2(a/b) + (c/4b)(\Delta x)^2]^{1/2}} \tag{14}$$

In terms of the plate properties Eq. (14) becomes

$$\Delta t \leq \left\{\frac{\rho(1 - \nu^2)/E}{2 + (1/12)(1 - \nu)\pi^2[1 + (3/2)(\Delta x/h)^2]}\right\}^{1/2} \Delta x \tag{15}$$

When a combination of time increment and space mesh sizes satisfies Eqs. (13) or (15), the calculation of the transient response of a moderately thick plate will be stable. However, Eq. (15) constitutes a more stringent condition than Eq. (13). It may be used for a rough estimate. When the amount of computing time becomes of primary concern, Eq. (13) should be used instead. For a constant space mesh size, Eqs. (13) and (15) indicate that when the plate thickness becomes smaller, the allowable time increment also becomes smaller. Although the classical plate theory can be deduced from the improved plate theory in an analytical way, Eqs. (13) and (15) impose a limitation on the applicability of the latter in the numerical solution of a thin plate under dynamic loads. This is due to the fact that for a relatively thin plate the time increment (for a constant space mesh size) must be kept sufficiently small to avoid any numerical instability. Therefore, as a result, a larger amount of computing time is required. It should be commented here that Eqs. (13) and (15) may also be used as the stability criterion for the calculation of transient response of moderately thick shells to dynamic loads. The reason for this possible extension has been given in Ref. 3.

Reduction to the Thin Plate Case

The stability condition for the finite-difference equation for a thin plate cannot be directly deduced from that for a moderately thick plate, i.e., Eqs. (13) and (15). This is because the latter has been derived by the consideration of the branch of roots of Eq. (9) represented by the case where the sign in front of $(\alpha^2 - \beta)^{1/2}$ in Eq. (11) is negative. This branch contains the higher frequencies of vibration of the discrete system which does not exist in the equation for a thin plate. These frequencies become infinite as the plate thickness approaches zero. To obtain the stability condition for a thin plate, one can examine the second branch of the roots which is given by the case where the sign in front of $(\alpha^2 - \beta)^{1/2}$ in Eq. (11) is positive. After making use of the fact that $\beta/\alpha^2 \ll 1$ for a thin plate and by using the same reasoning in obtaining Eq. (13), it can be shown that

$$\Delta t \leq \tfrac{1}{4}(\rho h/D)^{1/2}(\Delta x)^2 \tag{16}$$

Equation (16) is the same as Eq. (10) in Leech's paper.[1] In contrast to Eqs. (13) and (15), Eq. (16) indicates that for a constant space mesh size, the allowable time increment becomes larger as the plate thickness becomes smaller.

Effect of Transverse Shear or Rotary Inertia

It is interesting to note that when either the effect of transverse shear or rotatory inertia is included, the sufficient stability condition is identical to that for the thin plate. This result can be derived in a similar manner. The actual derivation will not be given here in the interest of brevity.

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Analysis of Coaxial Free Mixing Using the Turbulent Kinetic Energy Method

P. T. Harsha* and S. C. Lee†

ARO Inc., Arnold Air Force Station, Tenn.

Nomenclature

C = concentration of jet species
D = nozzle diameter
H = total enthalpy
k = turbulent kinetic energy per unit mass

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* Research Engineer, J-Division Special Projects Section, Engine Test Facility. Member AIAA.

† Consultant; Associate Professor, Department of Mechanical and Aerospace Engineering, University of Missouri at Rolla, Rolla, Mo. Member AIAA.